SCHOOL

Trial WACE Examination, 2011

Question/Answer Booklet

MATHEMATICS SPECIALIST 3C/3D

Section One: Calculator-free

SOLUTIONS

Student Number: In figures | | | | | | | | |

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time for this section: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	6	6	50	40	33
Section Two: Calculator-assumed	13	13	100	80	67
			Total	120	100

Instructions to candidates

1. The rules for the conduct of Western Australian external examinations are detailed in the *Year 12 Information Handbook 2011*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

3. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil**, except in diagrams.

Section One: Calculator-free **(40 Marks)**

This section has **six (6)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(6 marks)**

(a) Find the equation of the tangent to the curve $x^3 - 4xy + y^3 = 1$ at the point (1, -2).
(3 marks)

$$3x^2 - 4y - 4x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$ ✓

$$x = 1, y = -2$$

$$3 + 8 + (-4 + 12)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{11}{8}$$ ✓

$$y + 2 = -\frac{11}{8}(x - 1)$$

$$11x + 8y + 5 = 0$$ ✓

(b) Evaluate $\int_{\sqrt{e}}^{e} \frac{1}{x(\ln x)^2}dx$ using the substitution $u = \ln x$. (3 marks)

$$u = \ln x$$

$$du = \frac{dx}{x} \text{ and } x = \sqrt{e}, u = 0.5 \quad x = e, u = 1$$ ✓

$$\int_{0.5}^{1} u^{-2}du = \left[-\frac{1}{u}\right]_{0.5}^{1}$$ ✓

$$= -1 + 2$$

$$= 1$$ ✓

Question 2 **(5 marks)**

The rate at which a reservoir is filling is given by:

$V'(t) = 150t^{-\frac{1}{2}} + 10$ litres/second, where volume V is in litres and time t is in seconds.

If the reservoir held 3000 litres after 16 seconds, find the volume of water after a quarter of an hour.

$V'(t) = 150t^{-\frac{1}{2}} + 10$

$\int dv = \int \left(150t^{-\frac{1}{2}} + 10\right) dt$ ✓

$V = 300t^{\frac{1}{2}} + 10t + c$ ✓

When t=16, V=3000

$3000 = 300 \times 4 + 10 \times 16 + c$

c = 1640 ✓

$V = 300t^{\frac{1}{2}} + 10t + 1640$

$\frac{1}{4}\ hour = 900\ seconds$ ✓

V = 300×30+10×900+1640

V = 19640 litres ✓

Question 3 **(6 marks)**

The transformation matrix $M = \begin{bmatrix} -2 & 1 \\ a & b \end{bmatrix}$.

M represents a shear of factor k parallel to the y-axis followed by a rotation of 90° clockwise.

(a) Use properties of the two transformations to explain why $|M| = 1$ (1 mark)

Neither transformation changes the area of a shape and so the magnitude of the determinant of M must be 1. ✓

(b) Determine the values of a, b and k. (3 marks)

$$\begin{bmatrix} -2 & 1 \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

$$= \begin{bmatrix} k & 1 \\ -1 & 0 \end{bmatrix}$$

$a = -1$ ✓
$b = 0$ ✓
$k = -2$ ✓

(c) The point P is transformed by M to the point (8, 3). Determine the coordinates of P. (2 marks)

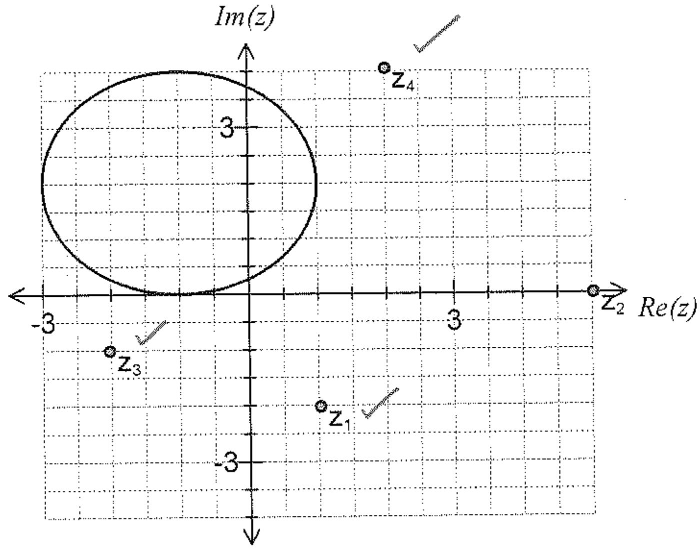
$$\begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 8 \\ 3 \end{bmatrix} ✓$$

$$= \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$P(-3, 2)$ ✓

Question 4 **(7 marks)**

The Argand diagram below shows the complex number z_1.



(a) On the same diagram plot and label the complex numbers given by (3 marks)

$$z_2 = z_1\bar{z}_1$$

$$z_3 = i^3 z_1$$

$$z_4 = 10z_1^{-1}$$

(b) On the same diagram sketch the region given by $|z + z_1| \le 2$. (2 marks)

centre ✓
radius ✓

(c) The locus of the complex number $z = x + iy$ satisfies the equation $|z+1| < |z-i|$.
Find the equation of the locus in the form $y > f(x)$ or $y < f(x)$. (2 marks)

The locus is the perpendicular bisector of the points (-1, 0) and (0, 1), on the side closer to the point (-1, 0).

Hence $y < -x$.

equation ✓
inequality ✓

Question 5 **(8 marks)**

(a) Use de Moivre's Theorem and the Binomial expansion to show that:

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$ (5 marks)

$$(\cos\theta + i\sin\theta)^4 = cos^4\theta + 4cos^3\theta(isin\theta) + 6cos^2\theta(isin\theta)^2 + 4cos\theta(isin\theta)^3 + (isin\theta)^4$$

$$= cos^4\theta - 6cos^2\theta sin^2\theta + sin^4\theta + i(4cos^3\theta sin\theta - 4cos\theta sin^3\theta)$$

Using dM's

$$(\cos\theta + i\sin\theta)^4 = cos(4\theta) + isin(4\theta)$$

Equating real parts:

$$cos(4\theta) = cos^4\theta - 6cos^2\theta sin^2\theta + sin^4\theta$$

$$= cos^4\theta - 6cos^2\theta(1 - cos^2\theta) + (1 - cos^2\theta)^2$$

$$= cos^4\theta - 6cos^2\theta + 6cos^4\theta + 1 - 2cos^2\theta + cos^4\theta$$

$$= 8cos^4\theta - 8cos^2\theta + 1$$

(b) Evaluate $\int_0^{\frac{\pi}{8}} (24\cos^4\theta - 24\cos^2\theta + 3)\,d\theta$ (3 marks)

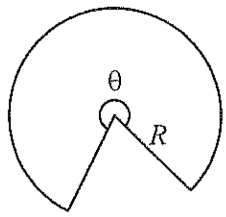
$$\int_0^{\frac{\pi}{8}} (24cos^4\theta - 24cos^2\theta + 3)d\theta = 3\int_0^{\frac{\pi}{8}} \big(cos(4\theta)\big)\,d\theta$$

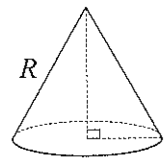
$$= \frac{3}{4}[sin(4\theta)]_0^{\frac{\pi}{8}}$$

$$= \frac{3}{4}$$

Question 6 **(8 marks)**

A minor sector of angle $2\pi - \theta$ is removed from a circular piece of paper of radius R. The two straight edges of the remaining major sector are pulled together to form a right circular cone, with a slant height of R.





(a) Show that the volume of the cone is given by $V = \dfrac{R^3\theta^2\sqrt{4\pi^2 - \theta^2}}{24\pi^2}$. (3 marks)

Let radius and height of cone be r and h..

Arc length of sector = circumference of cone base.

$$R\theta = 2\pi r \Rightarrow r = \frac{R\theta}{2\pi} \checkmark$$

$$h = \sqrt{R^2 - \left(\frac{R\theta}{2\pi}\right)^2} = \frac{R}{2\pi}\sqrt{4\pi^2 - \theta^2} \checkmark$$

$$V = \frac{1}{3}\pi \times \left(\frac{R\theta}{2\pi}\right)^2 \times \frac{R}{2\pi}\sqrt{4\pi^2 - \theta^2} \checkmark$$

$$V = \frac{R^3\theta^2\sqrt{4\pi^2 - \theta^2}}{24\pi^2}$$

(b) Find the value of θ which maximises the volume of cone. (5 marks)

$$u = \frac{R^3\theta^2}{24\pi^2} \qquad v = \sqrt{4\pi^2 - \theta^2}$$

$$u' = \frac{R^3\theta}{12\pi^2} \qquad v' = \frac{-\theta}{\sqrt{4\pi^2 - \theta^2}}$$

$$\frac{dV}{d\theta} = \frac{R^3\theta}{12\pi^2} \times \sqrt{4\pi^2 - \theta^2} + \frac{R^3\theta^2}{24\pi^2} \times \frac{-\theta}{\sqrt{4\pi^2 - \theta^2}} \checkmark\checkmark$$

$$= \frac{R^3\theta}{12\pi^2}\left(\sqrt{4\pi^2 - \theta^2} - \frac{\theta^2}{2\sqrt{4\pi^2 - \theta^2}}\right) \checkmark$$

$$= 0 \text{ when } \theta = 0 \text{ (Vmin) or } 2(4\pi^2 - \theta^2) - \theta^2 = 0 \checkmark$$

$$\Rightarrow 8\pi^2 = 3\theta^2$$

$$\Rightarrow \theta = \frac{2\sqrt{2}}{\sqrt{3}}\pi \checkmark \text{ (ignore -ve root as } 0 < \theta < 2\pi)$$